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Optimal Three Finger Grasps

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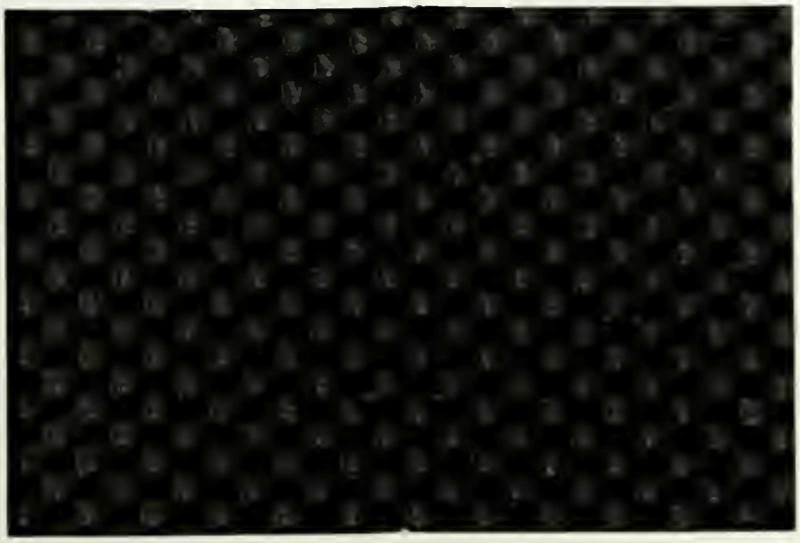

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Optimal Three Finger Grasps

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Abstract

We address the problem of optimal force distribution among three point fingers holding a planar object. A new scheme is presented which reduces the nonlinear optimization problem to a generalized eigenvalue problem which is easily solved. This scheme generalizes and simplifies results of Ji and Roth. The generalizations include all possible geometric arrangements and extensions to three dimensions and to the case of variable coefficients of friction.

1 Introduction

In this paper we consider the manipulation of objects using multifinger robot hands. In such manipulations it is important to analyze the distribution of forces among the gripping fingers. Because knowledge of coefficients of friction is only approximate it is desirable to find the finger forces which hold the object in equilibrium (or overcome an external force) while requiring the smallest coefficient of friction. (We assume a Coulomb friction model for the surface contacts.) That is, we seek finger forces for which the maximum of the angles θ_i $(i = 1, 2, 3)$ between each of them and the corresponding inward pointing normal is smallest.

We will prove here that, except for some separately treated special cases, the optimal equilibrium forces in the sense just described have the property that the θ_i's are all equal in absolute value. We also provide a procedure for computing these equilibrium forces, which reduces the (nonlinear) optimization problem to a generalized eigenvalue problem (which is easily solved).

We then extend our results to the case when the coefficients of friction at the contact points are different and to the three dimensional case. Corresponding geometric interpretations are given in both cases.

Our analysis is independent of the kinematic configuration of the fingers provided they can exert arbitrary forces within the friction cones. In particular, our results can be applied to such diverse multifinger manipulators as the Utah/MIT hand [7], the Salisbury hand [12], and NYU's Four Finger Manipulator [2].

In Section 2 we give a more precise formulation of the problem and introduce some notation. In Section 3 we compare our approach to others in the literature. In Sections 4 and 5 we prove our result on the geometric properties of the optimal equilibrium forces. In Section 6 we describe a procedure for finding those equilibrium forces. In Section 7 we present the generalizations.

2 Notation and definitions

We will assume throughout that we have three fingers and that the contact points between fingers and object are fixed. Moreover the contacts will be modeled as hard point contacts with friction. This means that each finger can transmit any force to the object through the contact (as long as it is within the cone of friction) but it can not transmit any torque. The relationship between the finger forces F_i ($i = 1, 2, 3$) and the resulting force and torque on the object f, τ can be described as follows (see Figure 1),

$$\sum_{i=1}^{3} F_i = f, \qquad \sum_{i=1}^{3} F_i \times r_i = \tau \tag{1}$$

where r_i is the position vector of the contact point p_i relative to the center of torque Q. In

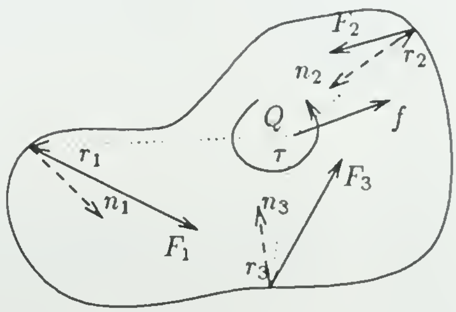

Figure 1: Finger forces.

the planar case the torque τ can be identified with a one dimensional quantity. Equation 1 can be rewritten as

$$GF = \begin{bmatrix} f \\ \tau \end{bmatrix} \tag{2}$$

where G is a 3×6 matrix in the 2-dimensional case (a 6×9 matrix in the 3-dimensional case), and $F = (F_1^T, F_2^T, F_3^T)^T$. Given an object force/torque pair (f, τ) to solve for the finger forces we use the pseudoinverse $G^+ = G^T(GG^T)^{-1}$ to get

$$F = G^+ \begin{bmatrix} f \\ \tau \end{bmatrix} + e \tag{3}$$

where e is an arbitrary vector in the right null space of G (NULL(G)). Since G has full rank, NULL(G) has dimension 3. That is, there are three independent parameters in the choice of the forces F_i.

We need to specify the vector e. This vector could be thought of as a vector of equilibrium forces. In particular the resulting torque should be zero. Therefore, except for the case when all forces are parallel, all three lines of action must intersect at the same point p. The point p accounts for two of the parameters in the choice of the F_i's, the third parameter being the norm of e (which represents the strength of the grip). The set of solutions resulting in parallel forces F_i can be parametrized using only two parameters (a unit vector and a magnitude).

Remark 2.1 If the F_i's are nonzero equilibrium finger forces (i.e. corresponding to $f = 0$ and $\tau = 0$ in (3)) and two of them are parallel then all three are parallel.

We will return to the case of parallel forces later. Now, we look at the problem of finding e from the point of view of finding p which minimizes $\theta_{\max} = \max_i |\theta_i(p)|$, where $\theta_i(p)$ is the angle that $p - p_i$ (and hence F_i) makes with the normal n_i to the object. To be more precise, $\theta_i(p)$ is the smaller angle between n_i and the line through p and p_i. Notice that $|\tan(\theta_{max})|$ is the smallest coefficient of friction necessary to achieve the given grasp.

The problem is one of minimizing the function $\max |\theta_i(p)|$ of two variables (the coordinates of p) over a region of the plane. However, because of geometric considerations which will be explained in Section 4 the search for a minimum can be restricted to a finite set, thus yielding a finite algorithm to find the optimal p. The several points p where the minimum might occur will be listed in Theorem 4.2. Each case will need to be treated separately.

One of the cases, however, merits special attention. That is when the minimum will be attained at a point p for which $|\theta_i(p)| = \theta$ for $i = 1,2,3$. To find such a minimum p we restate the problem as one of solving a generalized eigenvalue problem.

This method was implemented on NYU's Four Finger Manipulator. Experimental results can be found in [2,3].

We will need the following definitions. Given a set S, the interior of S, denoted $\overset{\circ}{S}$, is the set of all points p of S for which there is a disk (or a ball in the 3-D case) centered at p and completely contained in S. The closure of S, denoted $\overline{S}$, is the set of points p for which every disk (ball) centered at p intersects S. The boundary of S, denoted ∂S, is the set of those points of $\overline{S}$ which are not in $\overset{\circ}{S}$.

3 Comparison to other methods

Several results on the distribution of forces among several grasping fingers have been presented in the literature. The notion of *grip Jacobian* was presented in [12]. It corresponds to the matrix G introduced in the previous section. Instead of looking for optimal gripping forces Salisbury avoids the redundancy by specifying the magnitude of the components of the finger forces along the lines joining pairs of contact points. A similar transformation is presented in [6] but a method is given to avoid the inversion of the extended grip Jacobian.

Kerr and Roth [9] consider the general problem of selection of the internal forces (i.e. equilibrium forces) in both 2 and 3 dimensions. They simplify the problem into a linear programming one by approximating the 3-dimensional cones by piecewise linear pyramids. They combine all the constraints coming from friction cones and joint torque limits into a set of linear inequalities. These determine a constraint polygon in m dimensional space (where m depends on the number of fingers and joints). They then choose a point which is

furthest away from the boundary of this constraint polygon as the optimal internal forces. However, no geometric interpretation of the resulting finger forces is given.

Schwartz and Sharir [16] developed an algorithm for finding the finger forces, which generalizes to an arbitrary number of fingers in the plane. They study feasibility regions in force/torque space and present a complexity analysis of the algorithm. The approach is used to find optimal finger forces which overcome an arbitrary external generalized force on the object (and not simply equilibrium forces). Their method does not extend to 3-dimensions and does not provide an easy geometric interpretation of the finger forces.

The closest related work is that of Ji and Roth [8]. Using purely geometric reasonings they derive conditions for the internal forces (called equilibrium forces here) to minimize the dependence on contact friction. They only consider some generic geometric arrangements of the object normals at the contact points. By using a combination of topological and algebraic tools, instead of purely geometric ones, we are able to consider all possible configurations for the object normals and include in the analysis the possibility of parallel finger forces. Furthermore, our approach easily extends to the case of three dimensions and variable coefficients of friction.

An approach to optimal grasping which takes into account the task to be performed was presented in [11].

In a related paper, Yoshikawa and Nagai [18,19], studied the problem of decomposing the fingertip forces into their grasping and manipulating components. This led them to the study of the regions of the plane where a possible concurrency point p (as mentioned in the previous section) might lie. An analysis of grasping and manipulating ability of a given multifinger hand was also presented in [10].

We do not address here the problem of grasp synthesis, since we assume that the contact points are given. That problem has been addressed in the literature, most notably in [13] and [16] and to some extent in [11]. A knowledge based approach to grasp synthesis can be found in [17]. The problem of stability of a multifinger grasp was addressed in [14,15].

4 Geometric Analysis

We first examine the case of nonparallel forces. We write $\theta_i(p) = \text{ang}(p - p_i, n_i)$, and $c_i(p) = \tan(\theta_i(p))$. We will also assume that $c_i(p_i) = 0$. Now, for $\alpha > 0$, define the (two sided) cones (Figure 2)

$$K_i(\alpha) = \{q : |c_i(q)| \leq \tan(\alpha)\}.$$

That is, $K_i(\alpha)$ is the set of all points q for which the angle between $q - p_i$ and either n_i or $-n_i$ is less than or equal to α. For $\alpha \geq 0$ we will denote by $L_i^{\pm}$ the boundary lines of $K_i(\alpha)$ (see Figure 2): $L_i^{\pm}(\alpha) = \{q : c_i(q) = \pm\alpha\}$.

While the vector $p - p_i$ will determine the line of action of the force F_i, it will not determine the direction of F_i. The direction should be such that the F_i pushes into the object. Therefore we define

$$F_i = F_i(p) = \text{sgn}((p - p_i)^T n_i)(p - p_i)$$

where $\text{sgn}(x) = x/|x|$ if $x \neq 0$ and $\text{sgn}(0) = 0$. Notice that we will get $F_i = 0$ if $(p - p_i)^T n_i = 0$. In such a case, a force in the direction of $(p - p_i)$ would require an infinite coefficient of

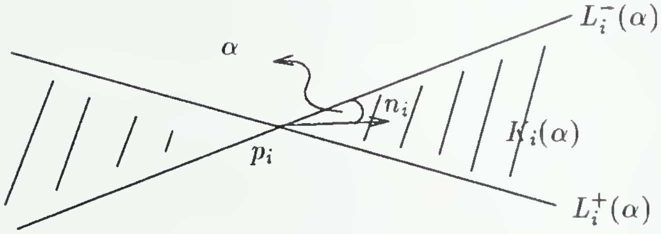

Figure 2: The cone $K_i(\alpha)$

friction and hence should not be considered. Moreover, we are only interested in equilibrium forces, so we want to find non-negative solutions $\varepsilon_i = \varepsilon_i(p)$ of

$$\sum_{i=1}^{3} \varepsilon_i F_i = 0 \qquad \sum_{i=1}^{3} \varepsilon_i = 1 \tag{4}$$

Define the set R^+ by

$$R^+ = \{p \in \Re^2 \colon \exists \varepsilon_i(p) \geq 0, i = 1, 2, 3, \text{ which solve } (4)\},$$

i.e. the set of feasible points p which result in equilibrium forces satisfying (4). Let R^0 be the set consisting of the three (not necessarily distinct) lines which pass through two of the contact points (the solid lines in Figure 3). (We are assuming all the contact points p_i are distinct, of course.) For $i = 1, 2, 3$, let $L_i^\perp = \{p : (p - p_i)^T n_i = 0\}$ be the tangent lines of the object through the points of contact (the dashed lines in Figure 3). The following lemma states more precisely what is implied in Figure 3.

Lemma 4.1 *The boundary of the set R^+ of feasible points p is contained in the union of the dashed lines and the solid lines in Figure 3, i.e.*

$$\partial R^+ \subset \bigcup_{i=1}^{3} L_i^\perp \cup R^0.$$

PROOF. First let p be such that there exist $\varepsilon_1(p)$, $\varepsilon_2(p)$, $\varepsilon_3(p)$ which solve the system (4) with $\varepsilon_{i_0}(p) = 0$ for some i_0. Then, the other two F_i's with $i \neq i_0$ are parallel (from the equilibrium equation (4)). Hence either one of them is zero and $p \in \bigcup_{i=1}^{3} L_i^\perp$ or p is on the line through the p_i's with $i \neq i_0$.

Note that (4) may be rewritten as

$$A(p) \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where

$$A(p) = \begin{bmatrix} F_1 & F_2 & F_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \sigma_1(p - p_1) & \sigma_2(p - p_2) & \sigma_3(p - p_3) \\ 1 & 1 & 1 \end{bmatrix}$$

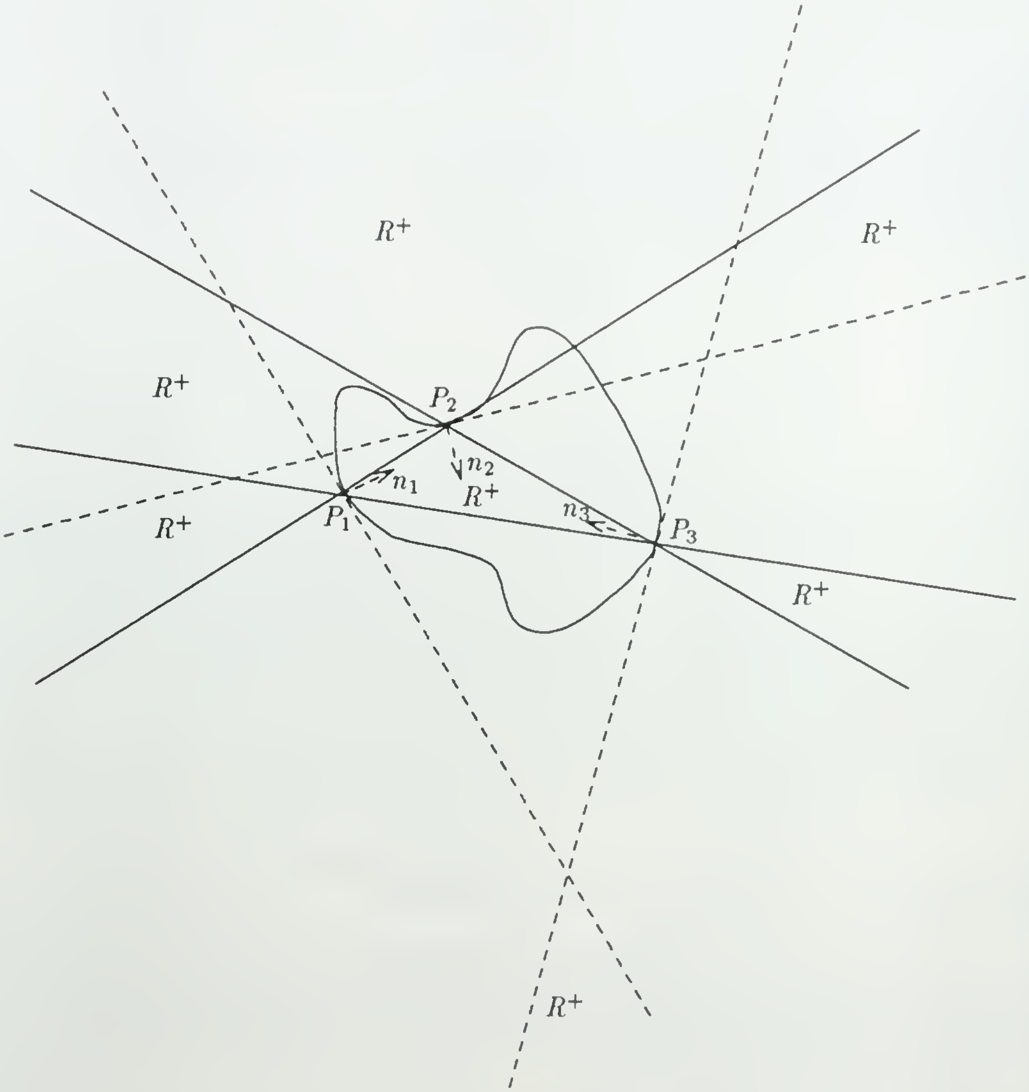

Figure 3: The region R^+.

and $\sigma_i = \mathrm{sgn}((p - p_i)^T n_i)$. We will show that if $A(p)$ is singular, then $p \in \bigcup_{i=1}^3 L_i^\perp \cup R^0$. In fact, if $A(p)$ is singular, then the affine space of solutions of (4) is at least one dimensional and must then intersect one of the coordinate planes $\varepsilon_i \equiv 0$. We can then reason as above to get $p \in \bigcup_{i=1}^3 L_i^\perp \cup R^0$.

Therefore, for $p \notin \bigcup_{i=1}^3 L_i^\perp \cup R^0$, the matrix $A(p)$ is nonsingular and the (uniquely defined) functions $\varepsilon_i(p)$ are continuous and have constant sign on each connected component of $\Re^2 \setminus \bigcup_{i=1}^3 L_i^\perp \cup R^0$. Then clearly, those points p for which the ε_i's are all positive are contained in the interior of R^+ and the remainder p's with nonzero ε_i's are contained in the interior of the complement of R^+. Hence, $\partial R^+ \subset \bigcup_{i=1}^3 L_i^\perp \cup R^0$. ∎

We are now ready for the main theorem.

Theorem 4.2 (Classification of optimal points) *Let p_i, $i = 1, 2, 3$ be three distinct points in the plane. For each i, let n_i be a unit vector with base at p_i. Let E be defined by,*

$$E = \{q \in \Re^2 : |c_1(q)| = |c_2(q)| = |c_3(q)|\}.$$

Let $p \in R^+$. Then, at least one of the following holds:

(a) $p \in R^0$,

(b) $p \in E$,

(c) there is p' such that

$$\max_{1 \le i \le 3} |c_i(p')| < \max_{1 \le i \le 3} |c_i(p)|$$

Remark 4.1 This theorem implies that the optimal p (those p minimizing $\max_i |c_i(p)|$) must lie in category a) or b).

Proof. Since $\max_i |c_i(q)| = \infty$ for any $q \in L_i^\perp$ for any i we need only search for optimal points p in R^0 and $\mathring{R}^+$ (see previous lemma). Since $p \in R^0$ is case (a), assume from now on that $p \in \mathring{R}^+$. Based on the ordering among $|c_1(p)|$, $|c_2(p)|$, and $|c_3(p)|$ we will classify p in one of the cases (a), (b), or (c).

By definition, if $|c_1(p)| = |c_2(p)| = |c_3(p)|$, then $p \in E$. We assume without loss of generality that $|c_1(p)| = \max_{1 \le i \le 3} |c_i(p)|$.

Consider first the case

$$|c_1(p)| > |c_2(p)|, \quad |c_1(p)| > |c_3(p)|.$$

Since $p \in \partial \mathring{K}_1(\theta_1(p))$ there exists a sequence $\{q_n\} \subset \mathring{K}_1(\theta_1(p))$ such that $q_n \to p$ (Figure 4).

Moreover, we may assume that the sequence $\{q_n\}$ is contained in $\mathring{R}^+$, since $p \in \mathring{R}^+$. Because $p \neq p_i$ for $i = 1, 2, 3$ the functions $c_i(p)$ are continuous near p. We can then find n_0 such that $|c_1(p)| > |c_2(q_{n_0})|$ and $|c_1(p)| > |c_3(q_{n_0})|$. Since we also have (by the definition of the cones) $|c_1(p)| > |c_1(q_{n_0})|$ we conclude that p is in case (c) (take $p' = q_{n_0}$).

Consider now the case

$$|c_1(p)| = |c_2(p)| > |c_3(p)|.$$

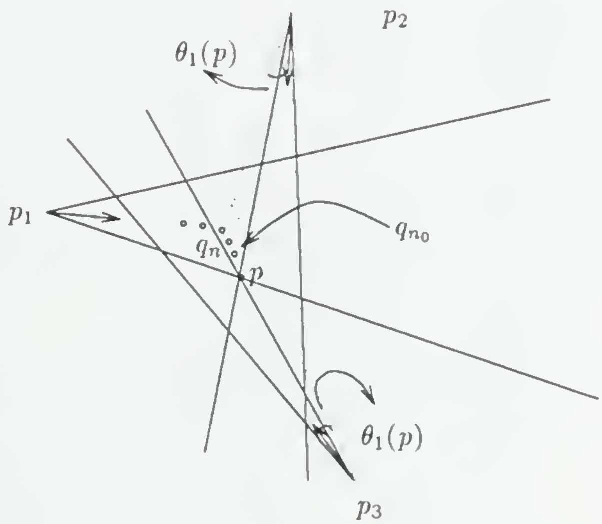

Figure 4: The point p is not optimal.

Say $p \in L_1 \cap L_2$, where

$$L_1 = L_1^+(\theta_1(p)) \text{ or } L_1 = L_1^-(\theta_1(p))$$

and

$$L_2 = L_2^+(\theta_2(p)) \text{ or } L_2 = L_2^-(\theta_2(p)).$$

If L_1 is not parallel to L_2, and since $p_1 \neq p_2$ then clearly (Figure 5)

$$p \in \partial \left(\mathring{K}_1(\theta_1(p)) \cap \mathring{K}_2(\theta_2(p)) \right)$$

Therefore, in this case we can imitate the previous reasoning by choosing the sequence $\{q_n\}$ in the intersection

$$\mathring{K}_1(\theta_1(p)) \cap \mathring{K}_2(\theta_2(p)) \cap \mathring{R}^+.$$

And so p again satisfies (c). Assume then L_1 and L_2 are parallel. Hence $L_1 = L_2 = L$. But then $p_1, p_2 \in L \subset R^0$ and hence $p \in R^0$, which is case (a). ∎

Remark 4.2 Cases (b) and (c) are not mutually exclusive.

We now carry out a similar analysis for the case of parallel forces.

As we said in the introduction, the directions of solutions which lead to parallel forces can be parametrized by the unit circle. Each unit vector p defines a set of parallel forces by

$$F_i = F_i(p) = \mathrm{sgn}(p^T n_i)p$$

So, for the case of parallel forces we want non-negative solutions of

$$\sum_{i=1}^{3} \varepsilon_i F_i = 0 \quad \sum_{i=1}^{3} \varepsilon_i F_i \times p_i = 0 \quad \sum_{i=1}^{3} \varepsilon_i = 1 \tag{5}$$

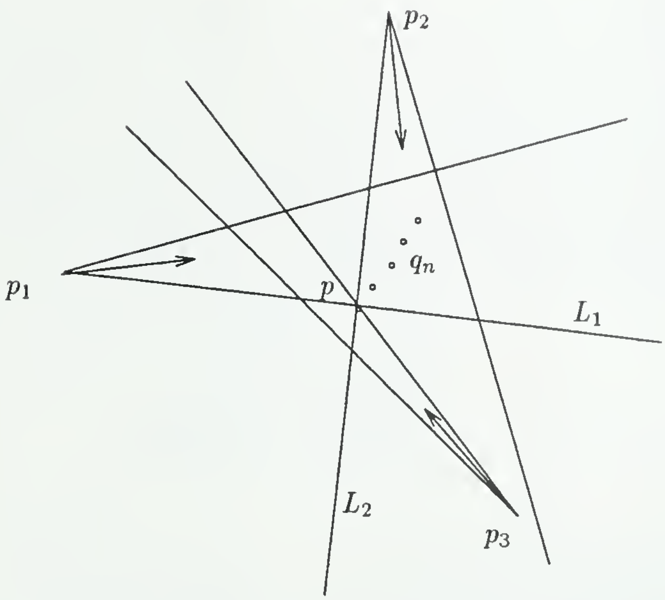

Figure 5: The point p is not optimal.

Analogously to the nonparallel case, we define a set $\widetilde{R}^+$ on the unit circle by (see Figure 6)

$$\widetilde{R}^+ = \{p\colon \|p\| = 1, \exists \varepsilon_i(p) \geq 0, i = 1,2,3, \text{ which solve } (5)\}.$$

Also let $\widetilde{R}^0$ be the set of lines through the origin parallel to the lines in R^0. We define corresponding $\widetilde{L}_i^{\perp}$ lines by

$$\widetilde{L}_i^{\perp} = \{p : p^T n_i = 0\}$$

For the case of parallel forces we then have the following theorem.

Theorem 4.3 *Let F_i, $i = 1,2,3$ be three parallel finger forces. Define $\widetilde{c}_i(F) = \tan(\operatorname{ang}(F, n_i))$ for $i = 1,2,3$. Let $\widetilde{E}$ be defined by*

$$\widetilde{E} = \{q \in \Re^2 : |\widetilde{c}_1(q)| = |\widetilde{c}_2(q)| = |\widetilde{c}_3(q)|\}.$$

Let $p \in \widetilde{R}^+$. Then, at least one of the following holds;

(a) $p \in \widetilde{R}^0$,

(b) $p \in \widetilde{E}$,

(c) there exists p such that $\max|\widetilde{c}_i(p)| < \max|\widetilde{c}_i(F_i)|$.

Proof. The proof is analogous to that of Lemma 4.1 and Theorem 4.2. Some slight differences arise from the fact that the two force equilibrium equations $\sum F_i = 0$ are not independent in this case.

The corresponding matrix $A(p)$ is now 4×3. Just as before, if $A(p)$ is not full rank, then there is a solution of (5) with $\varepsilon_{i_0} = 0$ for some i_0. Furthermore, assume one of the ε_i's is zero. To simplify the notation assume $\varepsilon_3 = 0$. Then from the force equilibrium equation

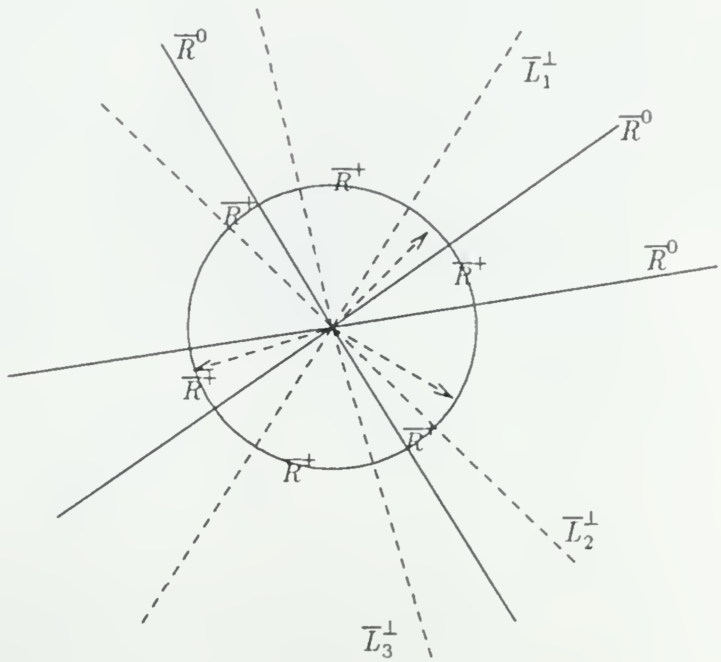

Figure 6: The set $\widetilde{R}^+$

we get $\varepsilon_2 F_2 = -\varepsilon_1 F_1$. Replacing into the torque equation we get $\varepsilon_1 F_1 \times (p_1 - p_2) = 0$. Since we assume $p_1 \neq p_2$ this implies $\varepsilon_1 F_1 = \lambda(p_1 - p_2)$ for some λ. If $\varepsilon_1 \neq 0$ then F_1 is parallel to $p_1 - p_2$ and hence $p \in \widetilde{R}^0$. If $\varepsilon_1 = 0$ then $\varepsilon_2 \neq 0$ (since $\sum \varepsilon_i = 1$). Also $\varepsilon_2 F_2 = 0$, so $F_2 = 0$ and hence $p \in \widetilde{L}_2^\perp$. Therefore, if $\varepsilon_i(p) = 0$ for some i then $p \in \cup \widetilde{L}_i^\perp \cup \widetilde{R}^0$.

If $p \notin \bigcup_{i=1}^3 \widetilde{L}_i^\perp \cup \widetilde{R}^0$, then $A(p)$ is full rank. Then $\varepsilon_i(p)$ is a continuous function of p for each i and hence has constant sign on each connected component of $\bigcup_{i=1}^3 \widetilde{L}_i^\perp \cup \widetilde{R}^0$. Just as in Lemma 4.1 we get

$$\partial \widetilde{R}^+ \subset \bigcup_{i=1}^{3} \widetilde{L}_i^\perp \cup \widetilde{R}^0.$$

The rest of the proof is completely analogous to that of Theorem 4.2. Simply translate the cones so their vertices are at the origin, intersect them with the unit circle and consider the boundaries and interiors relative to the unit circle. All the reasonings carry through. ∎

Remark 4.3 This theorem shows that the only case in which we need to study the possibility of nonzero parallel forces occurring is when all the $\widetilde{c}_i$'s are equal in absolute value. From this set we will choose a finite subset of forces minimizing $|\widetilde{c}_i|$.

5 Computing force vectors

We now need methods to find all points p which satisfy (a) or (b) of Theorem 4.2 or Theorem 4.3 and compute the finger forces associated with them. Note, however, that

these points do not necessarily belong to R^+ or $\tilde{R}^+$, requirements which must be checked later.

We will first list all the forces associated with points p arising from case (a) of both Theorem 4.2 and 4.3. While there might be infinitely many such points, they give rise to only finitely many lines of action for finger forces. Then we relate those points $p \in E$ and $p \in \tilde{E}$ to the solutions of a generalized eigenvalue problem (the same for both cases), which we analyze. That will give rise to a finite number of additional finger forces.

We start by analyzing case (a) of both theorems. First, let p equal one of the points p_i. For each of them we define force vectors F_i as follows.

Let $p = p_{i_0}$ and set

$$\begin{aligned} F_{i_0} &= n_{i_0} \\ F_j &= \mathrm{sgn}((p - p_j)^T n_j)(p - p_j) \qquad \text{for } j \neq i_0. \end{aligned}$$

(Note that $c_{i_0}(p_{i_0}) = 0$.)

Next consider $p \in R^0$ but not equal to any p_i. Since any point in R^0 (other than the p_i's themselves) will result in two of the finger forces being parallel, by Remark 2.1 and Theorem 4.3, the third force would have to be either zero or parallel to the other two. If the third force is nonzero and the contact points are not collinear, it will violate the torque equilibrium. So, if the contact points are not collinear, for each pair of them we choose nonzero finger forces parallel to the line through the pair for each of the two points and zero for the third. (The directions of the forces should be chosen so that the fingers push into the object, see Section 4.) If the contact points are collinear, for each of them we choose nonzero finger forces parallel to the line through all three of them. This concludes the consideration of case (a) from both Theorem 4.2 and 4.3.

We now study case (b) from both theorems. First we do the nonparallel case (i.e. Theorem 4.2). We transform the problem of finding the points $p \in E$ into a generalized eigenvalue problem as follows (see also [2,3]).

To simplify the notation assume $c_1(p) = c_2(p) = c_3(p) = -\mu$ (the other possibilities with different signs for $c_i(p)$ will be considered later). The number $|\mu|$ can be interpreted as the smallest coefficient of friction for which the grasp can be maintained. Let t_i be the unit tangent to the gripped object O at the i^{th} gripping point p_i. The point p satisfies

$$(p - p_i)^T(\mu n_i + t_i) = 0, \qquad i = 1, 2, 3 \tag{6}$$

Putting $\gamma_i = -p_i^T n_i$, $\eta_i = -p_i^T t_i$, Equation 6 can be written as

$$p^T(\mu n_i + t_i) = -(\mu\gamma_i + \eta_i). \tag{7}$$

Hence the vector $P = (p_1, p_2, 1)^T$ satisfies the equation

$$(M - \mu\overline{M})P = 0 \tag{8}$$

where M (resp. $\overline{M}$) is the matrix whose successive rows are (t_{ix}, t_{iy}, η_i), $i = 1, 2, 3$ (resp. $(-n_{ix}, -n_{iy}, -\gamma_i)$, $i = 1, 2, 3$). This is a generalized eigenvalue problem for the matrix pencil $M - \mu\overline{M}$, where μ is the eigenvalue and P is the eigenvector. To consider changes in the

signs of the $c_i(p)$'s, this eigenvalue equation must also be solved with the second and third rows of M changed (independently) to their negatives (four patterns in all).

Since it may happen that $(M-\mu\overline{M})$ is singular for all μ (such a pencil is called singular), any μ in the extended complex plane could be an eigenvalue. Nor does P need to be unique, even if $M-\mu\overline{M}$ is a regular pencil (i.e. not singular).

The general solution of (8) involves the Kronecker Canonical Form of the matrix pencil $M-\mu\overline{M}$. An algebraic description of this may be found in [5], and stable numerical algorithms for its practical solution may be found in [1,4].

Before proceeding to identify suitable μ's we consider the case of parallel forces (case (b) of Theorem 4.3). Let $p \in \widetilde{E}$. Then $F_i = \alpha_i p$ and $|\widetilde{c}_i(F_i)|$ is constant for $i = 1,2,3$. Assume as before that $\widetilde{c}_1(F_1) = \widetilde{c}_2(F_2) = \widetilde{c}_3(F_3) = -\mu$ We can then write

$$p^T(\mu n_i + t_i) = 0.$$

In turn this equation can be rewritten as

$$(M-\mu\overline{M})Q = 0$$

where $Q = [p^T, 0]^T$. Four cases will again be needed to allow for different signs of the $\widetilde{c}_i(F_i)$'s.

Therefore the geometric conditions $p \in E$, $p \in \widetilde{E}$ give rise to the eigenvalue problem (8).

Conversely, any real eigenvalue of (8) has a geometric interpretation. If the eigenvalue has an eigenvector P whose last coordinate is 1 then use the first two coordinates to get the point of concurrency p. If there is an eigenvector Q with last coordinate 0, then the first two coordinates define the direction of all three finger forces (which are therefore parallel).

We have then transformed the problem of identifying all forces arising from case (b) of Theorems 4.2 and 4.3 into solving the generalized eigenvalue problem (8). The remainder of the section is devoted to the study of the generalized eigenvalue problem. As we pointed out, there might be infinitely many eigenvalues. However, there is a finite subset of them which generates all possible forces which are candidates for an optimal grasp (among those arising from case (b)).

The following theorem summarizes the connection between eigenvectors and finger forces.

Theorem 5.1 *If p is such that $c_i(p) = -\mu$ for $i = 1,2,3$ then the vector $P = [p^T, 1]^T$ is an eigenvector of the matrix pencil $M-\mu\overline{M}$ ($\overline{M}$ if $\mu = \infty$). If $\widetilde{c}_i(F_i) = -\mu$ for $i = 1,2,3$ and there is a $q \neq 0$ such that $q^T F_i = 0$ (i.e., the F_i's are parallel) then the vector $P = [q^T, 0]^T$ is an eigenvector of the pencil $M-\mu\overline{M}$.*

Conversely, if $P = [x,y,1]^T$ is an eigenvector corresponding to a real eigenvalue μ then $p = [x,y]^T$ is a point such that $c_i(p) = -\mu$ for $i = 1,2,3$, and if $P = [x,y,0]^T$ is an eigenvector then $F_i = [-y,x]^T$ for $i = 1,2,3$ satisfy $\widetilde{c}_i(F_i) = -\mu$ for $i = 1,2,3$.

We describe the situation in more detail. If μ is an eigenvalue, then the dimension of the right null space of $M-\mu\overline{M}$ (the eigenspace) is either 1 or 2. The dimension 3 case is excluded because that would result in all tangent vectors t_i being multiples of their corresponding normal vectors n_i. If the dimension of the eigenspace is 1 there is a nonzero vector P which forms a basis for it. If the last coordinate of P is nonzero then there is exactly one eigenvector with a 1 as last coordinate and it should be used to determine p. If the last coordinate of P is zero then the last coordinate of any eigenvector is zero. In this

case any eigenvector determines the same direction for the parallel forces. If the dimension of the eigenspace is 2 then $M - \mu\overline{M}$ has rank 1 and so we can write

$$M - \mu\overline{M} = \begin{bmatrix} a^T & -p_1^T a \\ \alpha a^T & -\alpha p_1^T a \\ \beta a^T & -\beta p_1^T a \end{bmatrix}$$

for some nonzero 2-dimensional vector a and some nonzero scalars α, β. This implies that there is a basis $\{Q_1, Q_2\}$ of eigenvectors such that the last coordinate of Q_1 is 1 and the last coordinate of Q_2 is zero. In fact, $Q_2 = [b^T 0]^T$ is an eigenvector if and only if $a^T b = 0$. Since $a \neq 0$ there is only a one dimensional space of such b's. Hence the other eigenvector in the basis must have a nonzero last coordinate. As a consequence we also get that $p_1^T a = p_2^T a = p_3^T a$ and hence the points p_i are collinear. Therefore, the vectors of the form $Q_1 + \lambda Q_2$ give concurrence points that lie on the line that contains the p_i's and result in the same finger forces (all parallel to that line). Moreover, Q_2 determines forces which are also parallel to the line through the contact points. Summarizing, if the dimension of the eigenspace is 2 then any eigenvector determines finger forces which are parallel to the line passing through all three contact points.

As an example, the following proposition shows that a real eigenvalue always exists if the object is convex and the n_i's are not all on the same half plane.

Proposition 5.2 *Let A be a convex set. Let n_i, $i = 1,2,3$ be the inward normals at three points p_i, $i = 1,2,3$ on the boundary of A. Assume that there is no vector $x \neq 0$ for which the 3 dot products $x^T n_i$, $i = 1,2,3$ have the same sign. Then the matrix $\overline{M}$ is nonsingular and hence the pencil $M - \mu\overline{M}$ has at least one real eigenvalue.*

Proof. Notice first that the singularity of $\overline{M}$ is independent of the base point used to represent the p_i's. In fact, if B_1 and B_2 are two base points then the corresponding matrices $\overline{M}_1$ and $\overline{M}_2$ satisfy

$$\overline{M}_1 \begin{pmatrix} I & B_2 - B_1 \\ 0 & 1 \end{pmatrix} = \overline{M}_2$$

We may then assume that the base point is in the convex set. Hence, $p_i^T n_i \geq 0$ for all $i = 1,2,3$. Clearly we may further assume that $p_i^T n_i \neq 0$ for all i.

Assume there is $\lambda = (\lambda_1, \lambda_2, \lambda_3) \neq 0$ such that $\lambda\overline{M} = 0$. In particular,

$$\sum_{i=1}^{3} \lambda_i p_i^T n_i = 0$$

Therefore by our previous assumptions there exist two λ_i's with opposite signs. Say, $\lambda_1 < 0$ and $\lambda_2 > 0$.

Let $x \neq 0$ be a vector perpendicular to n_3. By the choice of λ we have

$$\sum_{i=1}^{3} \lambda_i x^T n_i = \lambda_1 x^T n_1 + \lambda_2 x^T n_2 = 0 \tag{9}$$

By hypothesis $x^T n_1$ and $x^T n_2$ have opposite signs. But then $\lambda_1 x^T n_1$ and $\lambda_2 x^T n_2$ would have the same sign, contradicting Equation 9. Therefore $\overline{M}$ is nonsingular. But then the

eigenvalues of the pencil $M - \mu\overline{M}$ are the same as the eigenvalues of the matrix $M\overline{M}^{-1}$. Since the polynomial $\det(\mu I - M\overline{M}^{-1}$ is of degree three it has at least one real root. ∎

In order to get a finite procedure we must analyze the case when the matrix pencil $M - \mu\overline{M}$ is singular, i.e. $\det(M - \mu\overline{M})$ is zero for all μ (and hence has infinitely many eigenvalues). As we said earlier, if the p_i's are collinear, then all the eigenvalues μ give rise to forces parallel to the line through the p_i's (which coincides with the set R^0). Such forces have already been included in the list of candidates and no new μ's need be considered. We may then assume that the p_i's are not collinear. As we showed earlier, this implies that the eigenspaces are all one dimensional. The following proposition will let us show that even in the singular case we need only examine finitely many eigenvalues. The proof is given in the appendix.

Proposition 5.3 *Assume that the p_i's are not collinear and that $M - \mu\overline{M}$ is a singular pencil. Then there exist polynomials $p_1(\mu)$, $p_2(\mu)$, and $d(\mu)$ of degree at most 2 (which can be computed using the Kronecker canonical form of $M - \mu\overline{M}$) such that the vector $P(\mu) = [p_1(\mu), p_2(\mu), d(\mu)]^T$ forms a basis of the right eigenspace for every μ.*

Assume now that μ is such that $d(\mu) \neq 0$. Divide each coordinate of the eigenvector by $d(\mu)$ to form the eigenvector $P(\mu) = [p_1(\mu)/d(\mu), p_2(\mu)/d(\mu), 1]^T = [p(\mu), 1]^T$. We only need to find the μ with smallest absolute value for which $p(\mu) \in R^+$. If μ_0 is such an eigenvalue, then, by Theorem 4.2, $\mu_0 = 0$ or $p(\mu_0) \in \partial R^+$ or $p(\mu) \to \infty$ as $\mu \to \mu_0$. Reasoning as in Theorem 4.2 in the second case we conclude that $p(\mu_0) \in R^0$. Therefore, such a case has already been discussed (as part of case (a)) and we do not need to find μ_0. The third case corresponds to $d(\mu_0) = 0$. But there are at most 2 such μ's and we include the corresponding finger forces in our list. If $d(\mu) \equiv 0$ then a similar reasoning eliminates all the eigenvalues except $\mu = 0$.

Summarizing, in the singular case we need only compute finitely many more μ's (at most 3 in fact).

6 Finding the equilibrium forces

We now have a finite set of finger forces F_i from which to choose the optimal one (in the sense of minimizing $\max|\tilde{c}_i(F_i)|$ and $\max|c_i(p)|$). We first find the appropriate equilibrium forces by solving the 3 by 3 system of linear equations

$$\sum_{i=1}^{3} \varepsilon_i F_i = 0, \quad \sum_{i=1}^{3} \varepsilon_i = 1. \tag{10}$$

in case the F_i's are not parallel, and the 4 by 3 system of linear equations

$$\begin{aligned} \sum_{i=1}^{3} \varepsilon_i F_i &= 0 \\ \sum_{i=1}^{3} \varepsilon_i F_i \times p_i &= 0 \\ \sum_{i=1}^{3} \varepsilon_i &= 1 \end{aligned} \tag{11}$$

in case all three forces are parallel. The equilibrium forces will exist if there are nonnegative ε_i's which satisfy the corresponding system of linear equations above.

In both cases the equilibrium forces are $\varepsilon_i F_i$ for $i = 1, 2, 3$.

To summarize we present the following algorithm for finding the optimal equilibrium finger forces.

Procedure

1. Solve the generalized eigenvalue problem (8).
2. Find the forces given by the p_i's, the forces parallel to the lines through the contacts, and the forces given by the generalized eigenvalue problem (selecting the appropriate ones if the matrix pencil is singular, see Section 5).
3. Find those sets of forces which give positive solutions to either Equation 11 or 10 (depending on whether the forces are all parallel or not). This can be checked by deciding if a point is in the region R^+ (see Figure 3) for the concurrent case, or on the arcs of $\widetilde{R}^+$ (see Figure 6) for the parallel case. If no such solution exists, then there is no grip possible and the procedure ends.
4. Among those sets for which positive solutions exist, choose the set for which $\max |\widetilde{c}_i(F)|$ is minimum.

Remark 6.1 In the end, the feasibility of applying the prescribed finger forces depends on whether they will stay within the cones of friction. However, the above procedure gives the best possible choice and hence if those are not feasible at least we know that the object can not be gripped using the given contact points.

7 Generalizations

We can extend our previous analysis in two different directions. First we will consider the 3-dimensional case; i.e. three fingers making point contact with a 3-dimensional object. It turns out that we can reduce our analysis to the 2-dimensional case. And in fact much of it will apply provided we make some changes in the definition of the functions $c_i(p)$. Second we will consider the 2-dimensional case when the coefficients of friction are different at the different contact points. In this case we need a priori information on the coefficients of friction but we still have an eigenvalue problem.

7.1 Three dimensional case

We will consider only the case of noncollinear contact points. Since we are looking for a set of equilibrium forces, all three of them must lie on the plane through the contact points. By restricting to this plane much of our 2-D analysis will apply provided we use a different $c_i(p)$. Notice that Theorem 4.2 holds for any $c_i(p) = c_i(\theta(p))$ which is monotone as a function of θ.

Assuming that the coefficients of friction at the contact points are the same, it seems natural to define the functions $c_i(p)$ in such a way that, when they are equal, the resulting angles between finger forces and normals are equal. (The normals are 3-dimensional vectors.)

We proceed as follows. Let n_i be the unit normal at the contact point p_i. (Unless otherwise stated i ranges from 1 to 3.) Let ω_i be the angle between n_i and the plane M that contains the points p_i (see Figure 7). Notice that ω_i does not depend on p.

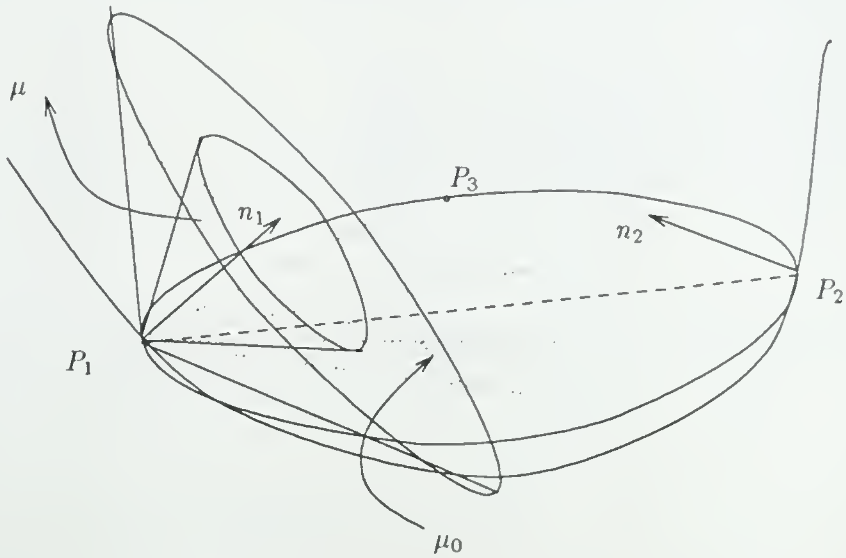

Figure 7: Intersection of three dimensional friction cones with plane through contact points.

Define

$$c_i(p) = \sqrt{\frac{\tan^2(\theta_i(p)) + \sin^2(\omega_i)}{\cos^2(\omega_i)}}$$

where $\theta_i(p)$ is defined as before. Since

$$c_i(p) = \tan(\mathrm{ang}(p - p_i, n_i))$$

this definition achieves the desired result. Now assume

$$c_1(p) = c_2(p) = c_3(p) = k.$$

Then,

$$\tan(\theta_i(p)) = \pm\sqrt{\cos^2(\omega_i)k^2 - \sin^2(\omega_i)}$$

Let $\mu_i(k) = -tan(\theta_i(p))$. Equation 6 now becomes

$$(p - p_i)^T(\mu_i(k)n_i + t_i) = 0, \tag{12}$$

which in turn can be transformed into (see Equation 8)

$$(M - D(k)\overline{M})P = 0 \tag{13}$$

where $D(k)$ is the 3×3 diagonal matrix with $D_{ii} = \mu_i(k)$, and M and $\overline{M}$ are as in Section 5. To obtain the optimal p we need to find the solution k of

$$\det(M - D(k)\overline{M}) = 0 \tag{14}$$

for which $\max_i |\mu_i(k)|$ is smallest. That is, we will have to find the roots of a (nonlinear) function of one real variable. While not as standard as an eigenvalue problem, this is still a very tractable problem.

7.2 Variable coefficients of friction

Another extension is to the 2-D case with variable coefficients of friction. In this case, we define the functions c_i's so that when they are equal the grasping forces will be inside the corresponding cone of friction in the same proportion. More precisely, assume that the coefficient of friction at p_i is β_i for $i = 1, 2, 3$ and define

$$c_i(p) = \frac{1}{\beta_i}\tan(\theta_i(p)).$$

Let $c_1(p) = c_2(p) = c_3(p) = k$ then

$$\beta_i k = \tan(\theta_i(p)).$$

Therefore Equation 8 becomes

$$(M - kD\overline{M})P = 0 \tag{15}$$

where D is the 3×3 diagonal matrix with $D_{ii} = \beta_i$ and M and $\overline{M}$ are as before. Notice that this is still a generalized eigenvalue problem where k is the eigenvalue and P is the eigenvector.

8 Conclusions

We describe a procedure to compute the grasping forces among three fingers making point contact with an object. Our analysis includes all possible geometric arrangements of point contacts and object normals. The method can be used in 2- and 3-dimensional cases and can handle variable coefficients of friction. For the 2-dimensional case with constant coefficients of friction we prove that, except for some special cases, the optimal grasping forces, in the sense of minimizing the dependence on friction, are those for which the angles with the corresponding normals are all equal (in absolute value).

A The matrix pencil $M - \lambda\overline{M}$

In this appendix we prove Proposition 5.3, which describes the set of solutions (μ, x) of the eigenproblem $(M - \mu\overline{M})x = 0$. In fact, we prove more, classifying all possible situations which can arise. This is done quite simply using the Kronecker Canonical Form (KCF) of the matrix pencil $M - \lambda\overline{M}$ (here λ is an indeterminate). To do this we need to define the KCF:

Theorem A.1 (Gantmacher) *Given an arbitrary (rectangular) pencil $A - \lambda B$, there exist nonsingular matrices P and Q such that $P(A - \lambda B)Q = S - \lambda T$ is block diagonal, where the diagonal blocks S_i and T_i may be of any one of the four forms $J_i(\lambda_0)$, N_i, L_i or L_i^T.*

$$J_i(\lambda_0) = \begin{bmatrix} \lambda_0 - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 - \lambda \end{bmatrix} \quad , \quad N_i = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix}$$

$J_i(\lambda_0)$ is simply an i by i Jordan block with finite eigenvalue λ_0. It has full rank unless $\lambda = \lambda_0$, in which case it has rank $i-1$ with right null vector $[1, 0, \ldots, 0]^T$ and left null vector $[0, \ldots, 0, 1]$. N_i is an i by i Jordan block for the eigenvalue at infinity. It has full rank unless "$\lambda = \infty$", in which case it has rank $i-1$ with the same right and left null vectors as above.

If the KCF only has blocks of the form $J_i(\lambda_0)$ and N_i, it is called regular. This happens if and only if $A - \lambda B$ is square and $\det(A - \lambda B)$ is not identically zero. Otherwise, the pencil is called singular and must contain at least one of the following two kinds of blocks:

$$L_i = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix} \quad , \quad L_i^T = \begin{bmatrix} 1 & & \\ -\lambda & \ddots & \\ & \ddots & 1 \\ & & -\lambda \end{bmatrix}$$

L_i is an i by $i+1$ block called a right singular block (L_0 is simply a zero column). It has a one dimension right null vector $[\lambda^i, \lambda^{i-1}, \ldots, 1]^T$ and no left null vector. L_i^T is an $i+1$ by i block called a left singular block (L_0^T is simply a zero row). It has a one dimension left null vector $[\lambda^i, \lambda^{i-1}, \ldots, 1]$ and no right null vector.

Thus, a pencil $A - \lambda B$ with singular blocks L_i or L_i^T can have a right null vector $P(\lambda)$ whose entries are polynomials in λ of degree at most i. Since an ,n by n pencil can have singular block no larger than L_{n-1} and L_{n-1}^T, it follows that the degree of $P(\lambda)$ can be at most $n-1$. This proves part of Proposition 5.3, which asserts that the degree of $P(\lambda)$ is at most 2 (since the pencil $M - \lambda\overline{M}$ is 3 by 3). It remains to show that such a $P(\lambda)$ exists. But the only kind of diagonal block in the KCF which does not have a right null vector is an L_i^T block. But since L_i^T has one more row than column, the KCF cannot contain only blocks of this form, since the original pencil is square. Therefore, a right null vector $P(\lambda)$ of degree at most 2 (it may be 0 or 1 as well) must exist. This completes the proof of Proposition 5.3.

To show that the complexity of dealing with the KCF is inherent in the problem, we prove a theorem enumerating all possible KCFs for pencils $M - \lambda\overline{M}$ arising in our grasping problem, and give examples of all different singular $M - \lambda\overline{M}$ which occur. To do this we introduce more notation. We say $A - \lambda B$ has an i dimensional regular part R_i if the sums of the dimensions of the regular blocks in its KCF is i. We say $A - \lambda B$ has *KCF*

$$m_0 \cdot L_0 \oplus m_1 \cdot L_1 \oplus \cdots \oplus n_0 \cdot L_0^T \oplus n_1 \cdot L_1^T \oplus \cdots \oplus \cdot R_j$$

if it has m_0 blocks of the form L_0 in its KCF, m_1 blocks of the form L_1 in its KCF, a j by j regular part, and so on.

Theorem A.2 *Let $M - \lambda\overline{M}$ be the pencil of Section 5. Then it must have one of the following four KCFs:*

1. R_3
2. $R_1 \oplus L_1 \oplus L_0^T$
3. $L_2 \oplus L_0^T$
4. $L_1 \oplus L_1^T$

Furthermore, there exist grasping configurations for which $M - \lambda\overline{M}$ exhibits each of these KCFs. (We have not been able to rule out case 4.)

PROOF. Since R_i is i by i, L_i is i by $i+1$, and L_i^T is $i+1$ by i, it is easy to list all possible KCFs of all 3 by 3 pencils. We simply enumerate all KCFs in the form (1) which form 3 by 3 pencils:

1. R_3
2. $L_0 \oplus L_0^T \oplus R_2$
3. $L_0 \oplus L_1^T \oplus R_1$
4. $2 \cdot L_0 \oplus 2 \cdot L_0^T \oplus R_1$
5. $L_1 \oplus L_0^T \oplus R_1$
6. $L_0 \oplus L_2^T$
7. $2 \cdot L_0 \oplus L_0^T \oplus L_1^T$
8. $3 \cdot L_0 \oplus 3 \cdot L_0^T$
9. $L_1 \oplus L_1^T$
10. $L_0 \oplus L_1 \oplus 2 \cdot L_0^T$
11. $L_2 \oplus L_0^T$

Next we will show that L_0 can not be in the KCF; this will eliminate all cases except the four in the statement of the theorem.

We argue by contradiction. Assume that L_0 is in the KCF. Then there is a constant nonzero vector $P = [x, y, z]^T$ which is a right null vector for $M - \lambda\overline{M}$ for all values of λ. Equating the i-th component of $(M - \lambda\overline{M})P$ to zero yields that the dot product of $[x - z \cdot p_{ix}, y - z \cdot p_{iy}]$ and $t_i - \lambda n_i$ is zero for all λ. (Here, $p_i = [p_{ix}, p_{iy}]^T$ is the i-th grasp point, t_i the i-th unit tangent vector and n_i the i-th normal vector.) Thus, $[x - z \cdot p_{ix}, y - z \cdot p_{iy}]$ must be identically zero. If $z \neq 0$, then $p_i = [x/z, y/z]^T$ for all i so that all the grasp points coincide, a contradiction. If $z = 0$, then x and y are zero so that P is zero, another contradiction.

It remains to exhibit grasp configurations with the KCF's 1) through 3). Case 1) is the generic case of a regular pencil; from Proposition 5.2 we see this occurs whenever we grasp a convex object "not all on one side".

Case 2) arises with parallel normals and collinear contact points as shown below: The contact points are all on the x-axis at $p_1 = -1$, $p_2 = 0$ and $p_3 = 1$. Thus

$$M - \lambda\overline{M} = \begin{bmatrix} -1 & -\lambda & -1 \\ 1 & \lambda & 0 \\ -1 & -\lambda & 1 \end{bmatrix}$$

Premultiplying $M - \lambda\overline{M}$ by P and postmultiplying it by Q where

$$P = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad and \quad Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yields

$$P \cdot (M - \lambda\overline{M}) \cdot Q = \begin{bmatrix} 1 & -\lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has KCF $L_1 \oplus N_0 \oplus L_0^T$. In particular, since $[\lambda, -1, 0]^T$ is a right null vector, there are parallel grasp forces which make equal angles with the normal for all possible angles. This is easily seen from Figure 8.

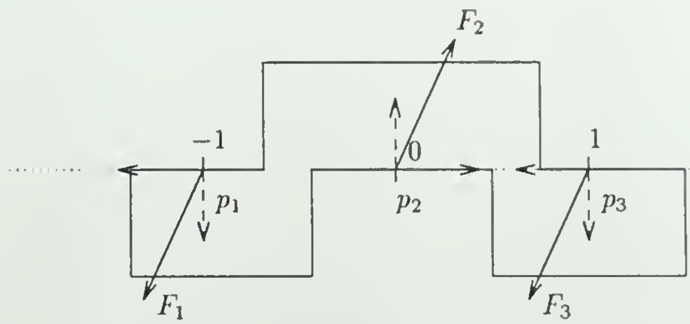

Figure 8: Case 2

Case 3) arises as follows. Let p_1, p_2 and p_3 be three arbitrary distinct points on the unit circle centered at the origin. Let the inward pointing normals in each case be parallel to the segment between p_i and $[-1, 0]$. Let the tangent vector in each case be parallel to the segment between p_i and $[1, 0]$. Since $[-1, 0]$ and $[1, 0]$ are at opposite ends of a diameter, elementary geometry shows that the normals and tangents constructed in this way are indeed orthogonal (see Figure 9).

We claim that $P(\lambda) = [\lambda^2 - 1, 2\lambda, \lambda^2 + 1]^T$ is a right null vector for all λ. If this is true, the KCF must contain an L_2 block, in which case the only other possible block in the KCF is L_0^T from dimensional considerations. Normalizing $P(\lambda)$, we get $[(\lambda^2 - 1)/(\lambda^2 + 1), 2\lambda/(\lambda^2 + 1), 1]^T = [x(\lambda), y(\lambda), 1]^T$ as a right null vector, so that all points of the form $[x(\lambda), y(\lambda)]$ are claimed to be possible centers of grasp. But it is easy to verify that $x^2(\lambda) + y^2(\lambda) = 1$, so that the claim is that all points p on the unit circle are possible centers of grasp. But this is

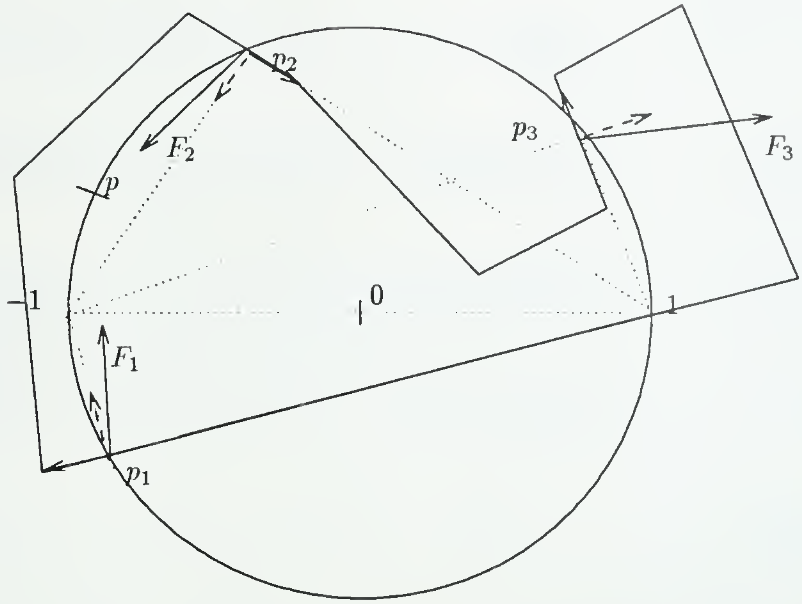

Figure 9: Case 3

easy to see, since from elementary geometry the angle between the segments from -1 to q and q to p is independent of q. In particular, the angle between the normals and the vectors from p_i to p are constants independent of the positions of the p_i; this proves the claim.

Case 4) arises as follows. Let all $t_i = (1,0)^T$ and $n_i = (0,-1)^T$ be parallel with $p_1 = (0,0)$, $p_2 = (-1,0)$ and $p_3 = (0,-1)$. Then

$$M - \lambda\overline{M} = \begin{bmatrix} 1 & -\lambda & 0 \\ 1 & -\lambda & 1 \\ 1 & -\lambda & -\lambda \end{bmatrix}$$

Premultiplying $M - \lambda\overline{M}$ by $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ yields

$$P(M - \lambda\overline{M}) = \begin{bmatrix} 1 & -\lambda & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\lambda \end{bmatrix}$$

which has $KCF = L_1 \oplus L_1^T$. ∎

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